

# Petrov types of slowly rotating fluid balls

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## Abstract

Circularly rotating axisymmetric perfect fluid space-times are investigated to second order in the small angular velocity. The conditions of various special Petrov types are solved in a comoving tetrad formalism. A number of theorems are stated on the possible Petrov types of various fluid models. It is shown that Petrov type II solutions must reduce to the de Sitter spacetime in the static limit. Two space-times with a physically satisfactory energy-momentum tensor are investigated in detail. For the rotating incompressible fluid, it is proven that the Petrov type cannot be D. The equation of the rotation function  $\omega$  can be solved for the Tolman type IV fluid in terms of quadratures. It is also shown that the rotating version of the Tolman IV space-time cannot be Petrov type D.

## 1 Introduction

Following Schwarzschild's discovery of the static incompressible interior solution in 1916, relentless efforts have been made to find a rotating generalization. Improved equations of state for the perfect fluid were imported in general relativity, leaving, however little clue for how to achieve this goal. Less than that, the supply of even the physically acceptable *nonrotating* solutions is scarce [1]. The common approach to solve the field equations of

the problem is to make some *a priori* assumption on the properties of the desired spacetime, *i. e.*, make an *ansatz*, and hope that solutions with the corresponding property do exist. More often than not, after considerable effort one gets a highly unphysical solution in this procedure, if any solution turns out to exist at all. A natural way to decide the acceptability of the various ansätze is to check whether or not they remain valid in the slowly rotating limit. Since static spherically symmetric spacetimes are either Petrov type D or 0, a widely studied ansatz is to assume that the rotating fluid also belongs to some special Petrov class. The aim of this paper is to investigate whether or not these classes contain physically acceptable perfect fluid spacetimes. It is natural to assume, that such configurations have a well behaving slow rotation limit.

We investigate axisymmetric stationary perfect fluid spacetimes in circular rotation, *i.e.*, with the fluid velocity vector lying in the plane of the timelike and angular Killing vectors,  $\partial/\partial t$  and  $\partial/\partial\varphi$  respectively. The metric of the spacetime is written in the form

$$ds^2 = \tilde{X}^2 dt^2 - \tilde{Y}^2 dr^2 - \tilde{Z}^2 \left[ d\vartheta^2 + \sin^2 \vartheta (d\varphi - \omega dt)^2 \right] , \quad (1)$$

where  $\tilde{X}$ ,  $\tilde{Y}$ ,  $\tilde{Z}$  and  $\omega$  are functions of the coordinates  $r$  and  $\vartheta$ . We choose the time-translation Killing vector  $\partial/\partial t$  such that it becomes asymptotically nonrotating at spacelike infinity. With this choice, the *angular velocity*  $\Omega$  of the fluid is defined by the components of the fluid velocity vector  $u^\mu$  as follows,

$$u^\varphi = \Omega u^t . \quad (2)$$

We further assume that the fluid is in *rigid rotation*, *i. e.*  $\Omega$  is a constant. This  $\Omega$  is the parameter which is small in the slow rotation limit, and following Hartle's work [2], we will expand quantities in powers of  $\Omega$ . For rigidly rotating space-times, a *comoving coordinate system*, where  $u^\varphi = 0$ , can be arranged by a linear transformation of the angular coordinate  $\varphi \longrightarrow \varphi - \Omega t$ . Since in this work we are interested only in the interior fluid region, we will perform our calculations in the comoving system.

We focus on the Petrov types that a slowly rotating fluid ball might have. We use a tetrad, constructed specifically for this task, representing the space-time to the desired order in the angular velocity  $\Omega$ . In this framework, definite statements can be made about the Petrov type of the field of a given order in  $\Omega$ , or about the impossibility of certain Petrov types *to any order* in  $\Omega$ . Nevertheless, the slow rotation approximation can be applied to give

information about numerous physical properties, even about the possible existence of an asymptotically flat exterior vacuum region [3].

Since the system behaves the same way under a reversal in the direction of rotation as under a reversal in the direction of time, when expanding the components of the metric (1) in powers of the angular velocity parameter  $\Omega$  one finds that  $\omega$  contains only odd powers, while  $\tilde{X}$ ,  $\tilde{Y}$  and  $\tilde{Z}$  contain solely even powers of  $\Omega$ . In this paper we are interested in effects of up to  $\Omega^2$  order, and hence we consider the function  $\omega$  to be proportional to the angular velocity parameter  $\Omega$ , while we allow  $\Omega^2$  terms in the diagonal components of the metric (1). The metric of a slowly rotating fluid ball can be written in the form [2]

$$ds^2 = (1 + 2h)X^2 dt^2 - (1 + 2m)Y^2 dr^2 - (1 + 2k)Z^2 \left[ d\vartheta^2 + \sin^2 \vartheta (d\varphi - \omega dt)^2 \right]. \quad (3)$$

Here the functions  $X$ ,  $Y$  and  $Z$  depend only on the radial coordinate  $r$ , determining the spherically symmetric basis solution, while  $\omega$ ,  $h$ ,  $m$  and  $k$  are functions of both  $r$  and  $\vartheta$ . The potential  $\omega$  is small to first order in the angular velocity  $\Omega$ , but  $h$ ,  $m$  and  $k$  are second order small quantities. There are two minor differences between our metric form (3) and the corresponding formula in the paper of Hartle [2]. The first is that we do not use the radial gauge  $Z = r$ , since that choice is technically inconvenient for certain spherically symmetric perfect fluid exact solutions. The second difference is that our definition of the second order small quantity  $m$  differs by a zeroth order factor. This choice is only to make our equations shorter, especially because the local mass function in the denominator of Hartle's definition takes a more complicated form in a general radial gauge  $Z \neq r$ .

Let us consider a freely falling observer, with velocity vector  $v^\mu$ , who has zero impact parameter and consequently zero angular momentum. In the coordinate system where  $\partial/\partial t$  corresponds to the timelike Killing vector which is nonrotating at spacelike infinity,  $v_\varphi = 0$ , and the function  $\omega$  agrees with the angular velocity  $v^\varphi/v^t = g^{\varphi t}/g^{tt} = -g_{\varphi t}/g_{\varphi\varphi}$  of this observer. Hence  $\omega$  can be interpreted as the rate of rotation of the local inertial frame with respect to the distant stars[2], or in other words, the cumulative dragging of initial frames[4]. In comoving coordinates, the rotation potential  $\omega$  represents the coordinate angular velocity of the fluid element at  $(r, \vartheta)$  measured by a freely falling observer to first order in  $\Omega$ .

We next briefly recapitulate Hartle's arguments leading to an unambigu-

ous choice of the coordinate system. For the full details, *cf.* [2]. We start from a known spherically symmetric perfect fluid solution, which is described by the functions  $X$ ,  $Y$  and  $Z$ . Calculating the field equations to *first order* in the small angular velocity parameter  $\Omega$ , we get only one independent relation [5] [6]. The  $(t, \varphi)$  component of Einstein equation gives a second order partial differential equation for  $\omega$ . In the comoving coordinate system this equation takes the form

$$\frac{X}{YZ^2} \frac{\partial}{\partial r} \left( \frac{Z^4}{XY} \frac{\partial \omega}{\partial r} \right) + 4 \frac{XZ}{Y} \left[ \frac{d}{dr} \left( \frac{1}{XY} \frac{dZ}{dr} \right) \right] \omega + \frac{1}{\sin^3 \vartheta} \frac{\partial}{\partial \vartheta} \left( \sin^3 \vartheta \frac{\partial \omega}{\partial \vartheta} \right) = 0 . \quad (4)$$

Expansion of (4) in vector spherical harmonics yields the angular behavior of the solution in the form

$$\omega = \sum_{l=1}^{\infty} \omega_l \left[ -\frac{1}{\sin \vartheta} \frac{dP_l(\cos \vartheta)}{d\vartheta} \right] , \quad (5)$$

where the functions  $\omega_l$  depend only on the radial coordinate  $r$ , and  $P_l$  is the Legendre polynomial of order  $l$ . The equations for the coefficients  $\omega_l$  with different values of  $l$  decouple. Taking into account the matching conditions at the surface of the fluid ball, one can show that for  $l > 1$  the functions  $\omega_l$  cannot be regular both at the center of the fluid and at infinity. It follows from the asymptotic flatness of the exterior spacetime region that  $\omega$  cannot depend on the angular coordinate  $\vartheta$ . Consequently, the rotation potential  $\omega$  is a function of the radial coordinate  $r$  alone, even in the fluid region, satisfying a second-order ordinary linear differential equation.

Since the expansion of  $\omega$  in the angular velocity parameter cannot contain  $\Omega^2$  terms, the solution of (4) will remain valid to second order as well. After solving the first order condition for the function  $\omega$ , proceeding to second order in the angular velocity parameter  $\Omega$ , the components of the Einstein equation give a coupled linear inhomogeneous system of partial differential equations for the functions  $h$ ,  $m$  and  $k$ . The inhomogeneous terms in the equations are proportional to  $\omega^2$  and its derivatives. The solution of this system can be written in the form of an expansion in spherical harmonics,

$$h = \sum_{l=0}^{\infty} h_l P_l(\cos \vartheta) , \quad (6)$$

and similarly for  $m$  and  $k$ . However, quantities with different  $l$  decouple, and the equations for  $l > 2$ , being homogeneous, do not include  $\omega$ . As a result,

all  $h_l$ ,  $m_l$  and  $k_l$  must vanish for  $l > 2$ , since otherwise they would correspond to a static but not spherically symmetric configuration in the  $\omega = 0$  case. Thus the second order small functions can be written in the form

$$\begin{aligned} h &= h_0 + h_2 P_2(\cos \vartheta) , \\ m &= m_0 + m_2 P_2(\cos \vartheta) , \\ k &= k_2 P_2(\cos \vartheta) , \end{aligned} \tag{7}$$

where  $h_0$ ,  $h_2$ ,  $m_0$ ,  $m_2$  and  $k_2$  are functions of  $r$ . The freedom in the choice of radial coordinate was used to set the  $\vartheta$  independent part of  $k$  to zero.

In Ref. [2], Hartle writes out the detailed form of the equations describing the second order rotational perturbations in the  $Z = r$  gauge, and gives a procedure to determine the binding energy, the baryon number change, and the ellipticity of the fluid surface. However, this procedure involves numerical integration of a system of ordinary differential equations. Instead, in our paper, we focus on what can be said about the general physical properties of slowly rotating fluid bodies by analytical methods, without trying to solve the perturbation equations. There is an extensive literature on numerical simulations of slowly rotating bodies for various types of fluids. These include neutron stars and supermassive stars [7], incompressible fluids [8], polytropes [2] and realistic neutron matter equations of state [9].

In Sec. 2., we establish a tetrad formalism for the Petrov classification. We also prove a theorem showing that physically realistic slowly rotating perfect fluid spacetimes cannot be Petrov type II. Case studies for various equations of state are presented in the rest of the paper. In Sec. 3., we establish a theorem on incompressible fluids. This theorem indicates that circularly rotating states should be found in the algebraically general class. For Tolman IV fluids, in Sec. 4., we find the rotation function  $\omega$  in terms of quadratures, and we show that these fluids cannot be Petrov type D.

The rotation function  $\omega$  can be written down in terms of elementary functions for the rotating Whittaker space-time. Although the equation of state of the Whittaker fluid implies that the density decreases towards the center of the fluid ball, this class merits special attention. Among the rotating states of the Whittaker fluid, there is the exactly known Petrov type D Wahlquist solution. The rotating Whittaker fluid is dealt with in [3] and [10].

## 2 The Petrov types

To quadratic order in the angular velocity parameter  $\Omega$ , the nonvanishing components of a comoving tetrad can be chosen for the metric (3) as

$$\begin{aligned} e_0^t &= \left(1 + \frac{1}{2}\omega^2 \frac{Z^2}{X^2} \sin^2 \vartheta - h\right) \frac{1}{X} , \\ e_1^r &= \frac{1-m}{Y} , \quad e_2^\vartheta = \frac{1-k}{Z} , \quad e_3^t = \omega \frac{Z}{X^2} \sin \vartheta , \\ e_3^\varphi &= \left(-1 + \frac{1}{2}\omega^2 \frac{Z^2}{X^2} \sin^2 \vartheta + k\right) \frac{1}{Z \sin \vartheta} . \end{aligned} \tag{8}$$

The correctness of these expressions to the required order can be shown by checking that  $(e_a^\mu e_b^\nu g_{\mu\nu}) = \text{diag}(1, -1, -1, -1)$  up to  $\Omega^2$  terms, where Roman and Greek labels denote tetrad and spacetime indices, respectively, and  $g_{\mu\nu}$  are the components of the metric (3).

Using the tetrad components (8), we will compute the Ricci rotation coefficients, the tetrad components of the Riemann, Weyl and Einstein tensors, and also some expressions, which are vanishing for certain Petrov types. All these quantities are polynomial expressions in the metric components  $g_{\mu\nu}$ , the tetrad vector components  $e_a^\mu$ , and their coordinate derivatives. Since the tetrad vector components will be multiplied only by terms which have a regular  $\Omega = 0$  limit, we get the correct results to second order in  $\Omega$  even if we use expressions for the tetrad, which are only correct to the same order.<sup>1</sup>

The electric and magnetic curvature components are defined in terms of the tetrad components of the Weyl tensor as follows [11]:

$$\begin{aligned} E_1 &= C_{1010} , & E_2 &= C_{2020} , & E_3 &= C_{1020} , \\ H_1 &= {}^*C_{1010} , & H_2 &= {}^*C_{2020} , & H_3 &= {}^*C_{1020} . \end{aligned}$$

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<sup>1</sup>An alternative approach would be to consider temporarily (3) to be valid to arbitrary order in the angular velocity  $\Omega$ , calculate the exact tetrad vector components, the Riemann tensor and other necessary quantities, and finally expand the result to second order in  $\Omega$ . This procedure gives exactly the same results as the simpler one, but because of the appearance of complicated denominator terms it would be a tedious work even for modern computer algebra systems. An even more cautious way of calculation, leading to the same results once again, would be to compute the coordinate components of the Riemann tensor, and take the tetrad components only in the end. However, this approach would have a similar problem with the components of the inverse metric  $g^{\mu\nu}$ , namely, whether or not one calculates them only to second order from the beginning.

Because of the symmetries of the configuration, these six components are the only independent components of the Weyl tensor. A Newman-Penrose frame is readily defined by  $\mathbf{l} = (e_0 + e_3)/\sqrt{2}$ ,  $\mathbf{n} = (e_0 - e_3)/\sqrt{2}$ ,  $\mathbf{m} = (e_1 + ie_2)/\sqrt{2}$ . The nonvanishing Weyl spinor components are

$$\Psi_0 = \frac{1}{2}[E_2 - E_1 - 2H_3 + i(H_1 - H_2 - 2E_3)] , \quad (9)$$

$$\Psi_2 = \frac{1}{2}[E_1 + E_2 - i(H_1 + H_2)] , \quad (10)$$

$$\Psi_4 = \frac{1}{2}[E_2 - E_1 + 2H_3 + i(H_1 - H_2 + 2E_3)] . \quad (11)$$

The Petrov type of the spacetime can be determined by studying the properties of the Weyl spinor components. If  $\Psi_0 = \Psi_2 = \Psi_4 = 0$  the spacetime is conformally flat, consequently it is the nonrotating interior Schwarzschild solution[12]. If  $\Psi_0 = \Psi_4 = 0$  but  $\Psi_2 \neq 0$ , one can apply a tetrad rotation around the vector  $\mathbf{l}$ , which makes  $\mathbf{m}$  into  $\mathbf{m} + a\mathbf{l}$ . Then the new nonzero Weyl spinor components become  $\tilde{\Psi}_4 = 6(a^*)^2\Psi_2$ ,  $\tilde{\Psi}_3 = 3a^*\Psi_2$  and  $\tilde{\Psi}_2 = \Psi_2$ , and the Petrov type is determined by the number of the distinct roots of the equation  $0 = \tilde{\Psi}_4 b^4 + 4\tilde{\Psi}_3 b^3 + 6\tilde{\Psi}_2 b^2 \equiv 6\Psi_2 b^2(a^*b + 1)^2$ . This equation has two double roots, which shows that the spacetime is of type D. If  $\Psi_4 \neq 0$  the Petrov type is determined by the number of the distinct roots of the algebraic equation

$$\Psi_4 b^4 + 6\Psi_2 b^2 + \Psi_0 = 0 \quad (12)$$

for the parameter  $b$ . If  $\Psi_4 = 0$  but  $\Psi_0 \neq 0$  one can apply a reversal in the  $e_3$  direction to interchange the frame vectors  $\mathbf{l}$  and  $\mathbf{n}$ , and consequently interchange  $\Psi_4$  and  $\Psi_0$  as well. Type III solutions are excluded because  $\Psi_3$  and  $\Psi_1$  vanish. The type is N if  $\Psi_0 = \Psi_2 = 0$  but  $\Psi_4 \neq 0$ . It has been shown in [11] that all axistationary type N perfect fluid solutions can be interpreted as vacuum solutions with a negative cosmological constant. In the case  $\Psi_4 \neq 0$  the Petrov type is II if and only if  $\Psi_0 = 0$  but  $\Psi_2 \neq 0$ . Assuming  $\Psi_4 \neq 0$ , the Petrov type is D if and only if  $9\Psi_2^2 = \Psi_0\Psi_4 \neq 0$ . For our generic comoving tetrad, the real and imaginary parts of the equation  $9\Psi_2^2 = \Psi_0\Psi_4$  yield

$$2E_1^2 + 5E_1E_2 + 2E_2^2 - E_3^2 - 2H_1^2 - 5H_1H_2 - 2H_2^2 + H_3^2 = 0 , \quad (13)$$

$$4E_1H_1 + 5E_1H_2 + 5E_2H_1 + 4E_2H_2 - 2E_3H_3 = 0 . \quad (14)$$

These conditions characterize slowly rotating fields. For finite angular velocities, the conditions for higher-order terms may not be met any more. By the

generic properties of the power series expansion of differentiable functions, however, any statement about the *nonexistence* of a certain type slowly rotating fluid will hold in the exact sense.

In the spherically symmetric limit, when  $\Omega = 0$ , it is easy to check that  $\Psi_0 = \Psi_4 = -3\Psi_2$ , independently of the equation of state. There are only two possible Petrov classes then. If all  $\Psi_i$  are zero, the metric is the conformally flat Petrov type 0 incompressible interior Schwarzschild solution. All other spherically symmetric perfect fluid spacetimes are Petrov type D, since  $9\Psi_2^2 = \Psi_0\Psi_4 \neq 0$  holds for them. This has important consequences for rotating Petrov N and Petrov II fluids, and also for the  $\Psi_0 = \Psi_4 = 0$  Petrov D subclass. Since  $\Psi_4$  or  $\Psi_0$  vanishes in all of these cases, in the  $\Omega = 0$  limit all the Weyl spinor components must go to zero, and consequently these spacetimes must reduce to the incompressible interior Schwarzschild solution in the nonrotating limit. This is already a severe limitation, but as we will see shortly, higher order conditions on the slowly rotating fluid state will pose further restrictions, which make these classes irrelevant for the study of rotating isolated bodies.

**Lemma 1:** Circularly and rigidly rotating perfect fluids with  $\Psi_0 = 0$  must reduce to the de Sitter space-time in the slow-rotation limit.

*Proof:* Using (9), the real and imaginary parts of the condition  $\Psi_0 = 0$  give

$$E_2 - E_1 - 2H_3 = 0 , \quad (15)$$

$$H_1 - H_2 - 2E_3 = 0 . \quad (16)$$

We can prove the lemma without assuming the existence of an asymptotically flat region, i. e. when  $\omega$  may also depend on  $\vartheta$ . We denote the tetrad components of the Einstein tensor by  $G_{ab}$ . As we have discussed in the previous paragraph, the spacetime must reduce to the incompressible interior Schwarzschild solution in the nonrotating limit, and hence the metric is of the form (3) with

$$\begin{aligned} X &= A - \cos r , \\ Y &= R , \\ Z &= R \sin r , \end{aligned} \quad (17)$$

where  $A$  and  $R$  are constants, satisfying  $R > 0$  and  $1 < A < 3$ . It is also possible to get this result directly by solving the zeroth order parts of the



equations (15) and  $G_{11} = G_{22}$ . A linear combination of the first order parts of  $G_{03} = 0$  and (15) yields a differential equation for  $\omega$  without  $\vartheta$  derivatives. The general solution of that is

$$\omega = \left( \frac{2A}{\cos(r) + 1} - 1 \right) f_1(\vartheta) + \left( \frac{2A}{\cos(r) - 1} - 1 \right) f_2(\vartheta) , \quad (18)$$

where  $f_1(\vartheta)$  and  $f_2(\vartheta)$  are some functions of the  $\vartheta$  coordinate. This  $\omega$  is regular at the center of the fluid  $r = 0$  only if  $f_2(\vartheta) = 0$  or if  $A = 0$ . Supposing that  $A$  is nonzero, substituting into (16) the leading  $\cos^2 r$  terms yield

$$A \left( \sin \vartheta \frac{df_1(\vartheta)}{d\vartheta} + 2 \cos \vartheta f_1(\vartheta) \right) = 0 . \quad (19)$$

The solution of the second factor is proportional to  $1/\sin^2 \vartheta$ , which is divergent at the rotation axis  $\vartheta = 0$ . Consequently we must have  $A = 0$ . Substituting back again to (15) and (16) we get that the derivative of  $f_1(\vartheta)$  must be zero, and consequently  $\omega$  is a constant. Since  $A = 0$ , this is the de Sitter space-time in a rotating coordinate system. The pressure  $p$  and density  $\mu$  are constants,  $p = -\mu = 3/R$ . Thus the equation of state violates the weak energy condition.

The Petrov type is II in the following two cases: if and only if  $\Psi_0 = 0$  but  $\Psi_2$  and  $\Psi_4$  are nonzero, or when  $\Psi_4 = 0$  but  $\Psi_2$  and  $\Psi_0$  are nonzero. In the  $\Psi_4 = 0$  case we change the components of  $e_3$  to  $-1$  times those in (8) in order to exchange  $\Psi_0$  with  $\Psi_4$ . From our Lemma, it follows that

**Theorem 1:** Circularly and rigidly rotating perfect fluids of Petrov type II must reduce to the de Sitter space-time in the slow-rotation limit.

Our results also show that a slowly rotating perfect fluid with an equation of state satisfying the weak energy condition cannot be of Petrov type II.

Lemma 1 also applies to the  $\Psi_4 = \Psi_0 = 0$  but  $\Psi_2 \neq 0$  Petrov type D subcase. This shows that all physically acceptable rotating Petrov type D solutions must be in the  $9\Psi_2^2 = \Psi_0\Psi_4 \neq 0$  class. Since this case seems to be too complicated for general investigation, we proceed with studying perfect fluids with specific equations of states.

### 3 Incompressible fluid

The Schwarzschild metric of a non-rotating incompressible perfect fluid ball is described by (17). The pressure and density are

$$\mu = \frac{3}{R^2} , \quad p = \frac{3 \cos r - A}{R^2(A - \cos r)} . \quad (20)$$

We assume the existence of an exterior asymptotically vacuum region, in which case  $\omega$  cannot depend on  $\theta$ . Calculating to second order in the small angular velocity parameter  $\Omega$ , the  $(t, \varphi)$  component (4) of the Einstein equations gives the condition

$$(A - \cos r) \sin r \frac{d^2 \omega}{dr^2} - (3 \cos^2 r - 4A \cos r + 1) \frac{d\omega}{dr} - 4A\omega \sin r = 0 . \quad (21)$$

There are also three other second-order field equations involving the functions  $h$ ,  $m$  and  $k$ .

Using the tetrad (8), we get that the vorticity is linear in the angular velocity parameter  $\Omega$ ,

$$\omega_1 = \frac{\cos \vartheta}{\cos r - A} \omega , \quad (22)$$

$$\omega_2 = \frac{\sin \vartheta}{2(A - \cos r)^2} \left[ 2\omega(A \cos r - 1) + \sin r (A - \cos r) \frac{d\omega}{dr} \right] . \quad (23)$$

The changes of all other rotational coefficients are small to second order.

The electric part of the Weyl tensor is quadratic in the angular velocity parameter  $\Omega$ , and the magnetic part has only linear terms,

$$H_1 = \frac{\cos \vartheta}{R(\cos r - A)} \frac{d\omega}{dr} , \quad H_2 = -\frac{1}{2} H_1 , \quad (24)$$

$$\begin{aligned} H_3 &= \frac{\sin \vartheta}{2R(A - \cos r)^2} \left[ \cos r(\cos r - A) \frac{d\omega}{dr} + 2A\omega \sin r \right] \\ &= -\frac{\omega_2 \cos r}{R \sin r} + \frac{\omega \sin \vartheta}{R \sin r (A - \cos r)} . \end{aligned} \quad (25)$$

In getting  $H_3$ , we used (21) to eliminate the second derivative of  $\omega$ .

Since the other subcase is excluded by Lemma 1, the Petrov type can be D if and only if Eqs. (13) and (14) hold. For the slowly rotating incompressible

fluid the only condition they give is  $H_3^2 - 2H_1^2 - 2H_2^2 - 5H_1H_2 = 0$ , which implies

$$\cos r(\cos r - A)\frac{d\omega}{dr} + 2\omega A \sin r = 0 . \quad (26)$$

The general solution of this equation is

$$\omega = C \left( \frac{A}{\cos r} - 1 \right)^2 , \quad (27)$$

where  $C$  is a constant. However, this  $\omega$  is not a solution of (21), and hence we have proven:

**Theorem 2:** A slowly and circularly rotating incompressible perfect fluid spacetime with an asymptotically flat vacuum exterior cannot be Petrov type D.

## 4 Tolman fluid

A spherically symmetric perfect fluid space-time has been given in [13]. The metric is described by

$$\begin{aligned} X^2 &= \frac{B^2(r^2 + A^2)}{A^2} , \\ Y^2 &= \frac{A^2 + 2r^2}{(R^2 - r^2)(A^2 + r^2)} R^2 , \\ Z &= r , \end{aligned} \quad (28)$$

where  $A$ ,  $B$  and  $R$  are constants. The equation of state is quadratic in the pressure,

$$R^2(A^2 + 2R^2)\mu = 4R^4A^2p^2 + R^2(2R^2 + 13A^2)p + 6(R^2 + 2A^2) . \quad (29)$$

The metric has a pleasingly simple form and describes an isolated fluid body with a regular center, outwards decreasing density and pressure and subluminal sound speed for appropriately chosen parameters [1].

We assume the existence of an asymptotically flat exterior vacuum region, in which case the function  $\omega$  does not depend on  $\vartheta$ , and the  $(t, \varphi)$  component (4) of the Einstein equations gives

$$\begin{aligned} (r^2 - R^2)(2r^2 + A^2)\frac{d\omega^2}{dr^2} + \left(5A^2r - 4A^2\frac{R^2}{r} + 8r^3 - 6rR^2\right)\frac{d\omega}{dr} \\ + 4(2R^2 + A^2)\omega = 0 . \end{aligned} \quad (30)$$

The general solution of this is

$$\omega = \frac{\sqrt[4]{2r^2 + A^2}}{r^2 \sqrt[4]{R^2 - r^2}} \left[ C_1 \exp \left( \frac{1}{2} \int \frac{P_1 + 2\sqrt{P_2}}{P_3} dr \right) + C_2 \exp \left( \frac{1}{2} \int \frac{P_1 - 2\sqrt{P_2}}{P_3} dr \right) \right], \quad (31)$$

where  $C_1$  and  $C_2$  are constants and

$$P_1 = 8r^{10} + (10R^2 + 13A^2)r^8 + (6A^4 + 4A^2R^2 - 8R^4)r^6 + A^2(A^4 + 4A^2R^2 - 4R^4)r^4 + R^2A^4(A^2 - 6R^2)r^2 - 2A^6R^4 \quad (32)$$

$$P_2 = R^2r^6(8R^2 + A^2)(5R^2 + 4A^2)(R^2 - r^2)(2r^2 + A^2)^3 \quad (33)$$

$$P_3 = r(r^2 - R^2)(2r^2 + A^2)(r^6 + 4r^4R^2 + 2A^2r^4 + r^2A^4 + 2r^2A^2R^2 - A^4R^2). \quad (34)$$

Since the  $\Psi_0 = \Psi_4 = 0$  subcase is excluded by Lemma 1, the rotating fluid can be Petrov type D only if equations (13) and (14) are satisfied. Using the tetrad (8) we get that (14) holds identically to second order in the angular velocity parameter  $\Omega$ . After substituting for the second derivative of  $\omega$  from (30) the Petrov D condition (13) yields

$$(r^2 - R^2)^2(A^2 + 2r^2) \left( \frac{d\omega}{dr} \right)^2 + 4r(r^2 - R^2)(A^2 + 2R^2)\omega \frac{d\omega}{dr} + 4r^2 \frac{(A^2 + 2R^2)^2}{A^2 + 2r^2} \omega^2 + 6B^2R^2 \frac{(A^2 + 2R^2)(A^2 + 2r^2)}{A^2(r^2 + A^2)}(m_2 - h_2) = 0. \quad (35)$$

Unlike in the incompressible case, this condition now involves the second order small quantities  $m_2$  and  $h_2$ . The reason for the easier treatment of the interior Schwarzschild solution was that because of its conformal-flat nature, both the electric and magnetic parts of the Weyl tensor were vanishing in the nonrotating limit. To be able to decide about the Petrov type, in general, one has to consider those parts of the Einstein equations, which are second order small in the angular velocity parameter  $\Omega$ . There are three such equations for the Tolman IV fluid. The pressure isotropy condition  $G_{22} = G_{33}$  takes the form

$$A^2r^4(r^2 - R^2)(A^2 + 2r^2) \left( \frac{d\omega}{dr} \right)^2 - 4A^2r^4(A^2 + 2R^2)\omega^2 + 6B^2R^2(A^2 + 2r^2)^2(m_2 + h_2) = 0. \quad (36)$$

The condition  $G_{12} = 0$  gives

$$r(A^2 + r^2)\frac{d}{dr}(h_2 + k_2) - A^2h_2 - (A^2 + 2r^2)m_2 = 0 , \quad (37)$$

while the  $P_2(\cos \vartheta)$  part of the another pressure-isotropy condition  $G_{11} = G_{22}$  yields a more lengthy second order differential equation

$$\begin{aligned} & 3B^2r(r^2 - R^2)(A^2 + 2r^2) \left[ (A^2 + r^2)r\frac{d^2(k_2 + h_2)}{dr^2} - (A^2 + 2r^2)\frac{dm_2}{dr} \right] \\ & + 3B^2r(4r^6 + A^2r^4 - 2R^2r^4 + 2R^2A^2r^2 + A^4R^2)\frac{dh_2}{dr} \\ & + 3B^2r^3(A^2 + 2R^2)(A^2 + r^2)\frac{dk_2}{dr} \\ & - 12B^2R^2(A^2 + 2r^2)^2(h_2 + k_2) + A^2r^4(r^2 - R^2)(A^2 + 2r^2) \left( \frac{d\omega}{dr} \right)^2 = 0 . \end{aligned} \quad (38)$$

In the following, we use equations (35) and (36) to eliminate  $h_2$  and  $m_2$ , while we employ (30) to express the second derivative of the rotation potential  $\omega$ . Comparing Eq. (38) with the derivative of (37) we get another first order differential equation. Eliminating the first derivative of  $k_2$  by (37), we get an equation that can be solved algebraically for  $k_2$ . Substituting  $k_2$  back to (37) again, we obtain an equation containing only  $\omega$ . Taking the  $r$  derivative of this equation, we readily find that it is not consistent with (30). Hence we have proven

**Theorem 3:** A rotating perfect fluid spacetime that reduces to the Tolman IV solution in the static limit and can be matched to an asymptotically flat vacuum exterior cannot be Petrov type D.

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